Beginning and Intermediate Algebra

Chapter 4: Systems of Equations

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Chapter 4: Systems of Equations

4.1

Systems of Equations - Graphing

We have solved problems like $3x - 4 = 11$ by adding 4 to both sides and then dividing by 3 (solution is $x = 5$). We also have methods to solve equations with more than one variable in them. It turns out that to solve for more than one variable we will need the same number of equations as variables. For example, to solve for two variables such as x and y we will need two equations. When we have several equations we are using to solve, we call the equations a **system of equations**. When solving a system of equations we are looking for a solution that works in both equation. This solution is usually given as an ordered pair (x, y). The following example illustrates a solution working in both equations

Example 1.

Show (2,1) is the solution to the system $\begin{array}{l} 3x - y = 5 \\ x + y = 3 \end{array}$

$$\begin{array}{rl}
(2,1) & \text{Identify } x \text{ and } y \text{ from the orderd pair} \\
x = 2, y = 1 & \text{Plug these values into each equation} \\
 & \\
3(2) - (1) = 5 & \text{First equation} \\
6 - 1 = 5 & \text{Evaluate} \\
5 = 5 & \text{True} \\
 & \\
(2) + (1) = 3 & \text{Second equation, evaluate} \\
3 = 3 & \text{True}
\end{array}$$

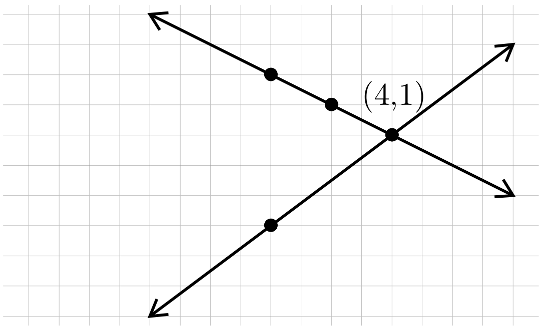
As we found a true statement for both equations we know (2,1) is the solution to the system. It is in fact the only combination of numbers that works in both equations. In this lesson we will be working to find this point given the equations. It seems to follow that if we use points to describe the solution, we can use graphs to find the solutions.

If the graph of a line is a picture of all the solutions, we can graph two lines on the same coordinate plane to see the solutions of both equations. We are interested in the point that is a solution for both lines, this would be where the lines intersect! If we can find the intersection of the lines we have found the solution that works in both equations.

Example 2.

$$\begin{aligned} y &= -\frac{1}{2}x + 3 \\ y &= \frac{3}{4}x - 2 \end{aligned} \qquad \text{To graph we identify slopes and } y-\text{intercepts}$$

$$\begin{aligned} \text{First: } m &= -\frac{1}{2}, b = 3 \\ \text{Second: } m &= \frac{3}{4}, b = -2 \end{aligned} \qquad \text{Now we can graph both lines on the same plane.}$$



To graph each equations, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots

Remember a negative slope is downhill!

Find the intersection point, (4,1)

(4,1) Our Solution

Often our equations won't be in slope-intercept form and we will have to solve both equations for y first so we can idenfity the slope and y-intercept.

Example 3.

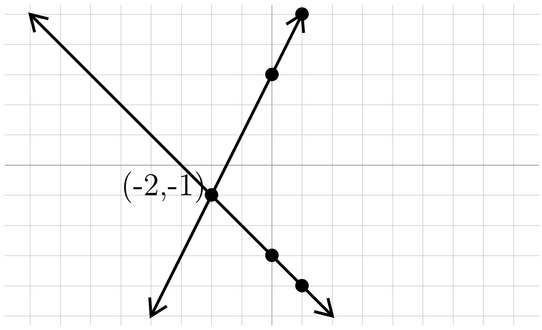
$$\begin{aligned} 6x - 3y &= -9 \\ 2x + 2y &= -6 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{rcl} 6x - 3y = -9 & 2x + 2y = -6 & \\ \underline{-6x \qquad -6x} & \underline{-2x \qquad -2x} & \text{Subtract } x \text{ terms} \\ -3y = -6x - 9 & 2y = -2x - 6 & \text{Put } x \text{ terms first} \\ \overline{-3} \quad \overline{-3} \; \overline{-3} & \overline{2} \quad \overline{2} \; \overline{2} & \text{Divide by coefficient of } y \\ y = 2x + 3 & y = -x - 3 & \text{Idenfity slope and } y-\text{intercepts} \end{array}$$

First: $m = \frac{2}{1}, b = 3$

Second: $m = -\frac{1}{1}, b = -3$

Now we can graph both lines on the same plane



To graph each equations, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots

Remember a negative slope is downhill!

Find the intersection point, $(-2, -1)$

$(-2, -1)$ Our Solution

As we are graphing our lines, it is possible to have one of two unexpected results. These are shown and discussed in the next two example.

Example 4.

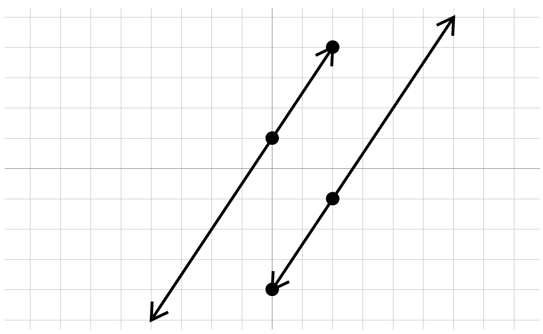
$$y = \frac{3}{2}x - 4$$
$$y = \frac{3}{2}x + 1$$

Identify slope and $y-$intercept of each equation

First: $m = \frac{3}{2}, b = -4$

Second: $m = \frac{3}{2}, b = 1$

Now we can graph both equations on the same plane



To graph each equations, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

The two lines do not intersect! They are parallel! If the lines do not intersect we know that there is no point that works in both equations, there is no solution

$\varnothing$ No Solution

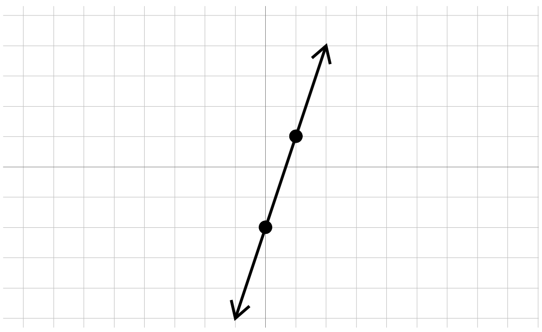
We also could have noticed that both lines had the same slope. Remembering that parallel lines have the same slope we would have known there was no solution even without having to graph the lines.

Example 5.

$$\begin{aligned} 2x - 6y &= 12 \\ 3x - 9y &= 18 \end{aligned} \qquad \text{Solve each equation for } y$$

$$\begin{array}{rr|l}
2x - 6y = 12 & 3x - 9y = 18 & \\
\underline{-2x \qquad -2x} & \underline{-3x \qquad -3x} & \text{Subtract } x \text{ terms} \\
-6y = -2x + 12 & -9y = -3x + 18 & \text{Put } x \text{ terms first} \\
\overline{-6} \quad \overline{-6} \quad \overline{-6} & \overline{-9} \quad \overline{-9} \quad \overline{-9} & \text{Divide by coefficient of } y \\
y = \frac{1}{3}x - 2 & y = \frac{1}{3}x - 2 & \text{Identify the slopes and } y - \text{intercepts}
\end{array}$$

$$\begin{aligned} &\text{First: } m = \frac{1}{3}, b = -2 \\ &\text{Second: } m = \frac{1}{3}, b = -2 \end{aligned} \qquad \text{Now we can graph both equations together}$$



To graph each equations, we start at the y-intercept and use the slope $\frac{\text{rise}}{\text{run}}$ to get the next point and connect the dots.

Both equations are the same line! As one line is directly on top of the other line, we can say that the lines “intersect” at all the points! Here we say we have infinite solutions

Once we had both equations in slope-intercept form we could have noticed that the equations were the same. At this point we could have stated that there are infinite solutions without having to go through the work of graphing the equations.

Practice - Graphing

Solve each equation by graphing.

1) $y = -x + 1$
$y = -5x - 3$

2) $y = -\frac{5}{4}x - 2$
$y = -\frac{1}{4}x + 2$

3) $y = -3$
$y = -x - 4$

4) $y = -x - 2$
$y = \frac{2}{3}x + 3$

5) $y = -\frac{3}{4}x + 1$
$y = -\frac{3}{4}x + 2$

6) $y = 2x + 2$
$y = -x - 4$

7) $y = \frac{1}{3}x + 2$
$y = -\frac{5}{3}x - 4$

8) $y = 2x - 4$
$y = \frac{1}{2}x + 2$

9) $y = \frac{5}{3}x + 4$
$y = -\frac{2}{3}x - 3$

10) $y = \frac{1}{2}x + 4$
$y = \frac{1}{2}x + 1$

11) $x + 3y = -9$
$5x + 3y = 3$

12) $x + 4y = -12$
$2x + y = 4$

13) $x - y = 4$
$2x + y = -1$

14) $6x + y = -3$
$x + y = 2$

15) $2x + 3y = -6$
$2x + y = 2$

16) $3x + 2y = 2$
$3x + 2y = -6$

17) $2x + y = 2$
$x - y = 4$

18) $x + 2y = 6$
$5x - 4y = 16$

19) $2x + y = -2$
$x + 3y = 9$

20) $x - y = 3$
$5x + 2y = 8$

21) $0 = -6x - 9y + 36$
$12 = 6x - 3y$

22) $-2y + x = 4$
$2 = -x + \frac{1}{2}y$

23) $2x - y = 1$
$0 = -2x - y - 3$

24) $-2y = -4 - x$
$-2y = -5x + 4$

25) $3 + y = -x$
$-4 - 6x = -y$

26) $16 = -x - 4y$
$-2x = -4 - 4y$

27) $-y + 7x = 4$
$-y - 3 + 7x = 0$

28) $-4 + y = x$
$x + 2 = -y$

29) $-12 + x = 4y$
$12 - 5x = 4y$

30) $-5x + 1 = -y$
$-y + x = -3$

4.2

Systems of Equations - Substitution

When solving a system by graphing has several limintations. First it requires the graph be perfectly drawn, if the lines aren't straight we may arrive at the wrong answer. Second, graphing is not a great method to use if the answer is really large, over 100 for example, or if the answer is a decimal that that graph won't help us find, 3.2134 for example. For these reasons we will rarely use graphing to solve our systems. Instead an algebraic approach will be used instead.

The first algebraic approach is called substitution. We will build the concepts of substitution through several examlpe, then end with a five-step process to solve problems using this method.

Example 6.

$x = 5$ $y = 2x - 3$	We already know $x = 5$, substitute this into the other equation
$y = 2(\mathbf{5}) - 3$	Evaluate, multiply first
$y = 10 - 3$	Subtract
$y = 7$	We now also have y
$(5, 7)$	Our Solution

When we know what one varaible equals we can plug that value (or expression) in for the variable in the other equation. It is very important that when we substitute, the substituted value goes in parenthesis. The reason for this is shown in the next example.

Example 7.

$2x - 3y = 7$ $y = 3x - 7$	We know $y = 3x - 7$, substitute this into the other equation
$2x - 3(\boldsymbol{3x - 7}) = 7$	Solve this equation, distributing -3 first

$$
\begin{array}{rl}
2x - 9x + 21 = 7 & \text{Combine like terms } 2x - 9x \\
-7x + 21 = 7 & \text{Subtract } 21 \\
\underline{\mathbf{-21 \quad -21}} & \\
-7x = -14 & \text{Divide by } -7 \\
\overline{\mathbf{-7}} \quad \overline{\mathbf{-7}} & \\
x = 2 & \text{We now have our } x\text{, plug into the } y = \text{ equation to find } y \\
y = 3(\mathbf{2}) - 7 & \text{Evaluate, multiply first} \\
y = 6 - 7 & \text{Subtract} \\
y = -1 & \text{We now also have } y \\
(2, -1) & \text{Our Solution}
\end{array}
$$

By using the entire expression $3x - 7$ to replace y in the other equation we were able to reduce the system to a single linear equation which we can easily solve for our first variable. However, the lone variable (a variable without a coefficient) is not always alone on one side of the equation. If this happens we can isolate it by solving for the lone variable.

Example 8.

$$
\begin{array}{rl}
\begin{array}{r} 3x + 2y = 1 \\ \boldsymbol{x} - 5y = 6 \end{array} & \text{Lone variable is } x\text{, isolate by adding } 5y \text{ to both sides.} \\
\underline{\boldsymbol{+5y + 5y}} & \\
x = 6 + 5y & \text{Substitute this into the untouched equation} \\
3(\boldsymbol{6 + 5y}) + 2y = 1 & \text{Solve this equation, distributing 3 first} \\
18 + 15y + 2y = 1 & \text{Combine like terms } 15y + 2y \\
18 + 17y = 1 & \text{Subtract 18 from both sides} \\
\underline{\mathbf{-18 \qquad -18}} & \\
17y = -17 & \text{Divide both sides by 17} \\
\overline{\mathbf{17}} \quad \overline{\mathbf{17}} & \\
y = -1 & \text{We have our } y\text{, plug this into the } x = \text{ equation to find } x \\
x = 6 + 5(\mathbf{-1}) & \text{Evaluate, multiply first} \\
x = 6 - 5 & \text{Subtract} \\
x = 1 & \text{We now also have } x \\
(1, -1) & \text{Our Solution}
\end{array}
$$

The process in the previous example is how we will solve problems using substitution. This process is described and illustrated in the following table which lists the five steps to solving by substitution.

Problem	$4x - 2y = 2$ $2x + y = -5$
1. Find the lone varaible	Second Equation, y $2x + \boldsymbol{y} = -5$
2. Solve for the lone variable	$-2x \quad -2x$ $\boldsymbol{y = -5 - 2x}$
3. Substitute into the untouched equation	$4x - 2(\boldsymbol{-5 - 2x}) = 2$
4. Solve	$4x + 10 + 4x = 2$ $8x + 10 = 2$ $\underline{-10 \ -10}$ $8x = -8$ $\overline{8} \quad \overline{8}$ $\boldsymbol{x = -1}$
5. Plug into lone variable equation and evaluate	$y = -5 - 2(\boldsymbol{-1})$ $y = -5 + 2$ $\boldsymbol{y = -3}$
Solution	$(-1, -3)$

Sometimes we have several lone variables in a problem. In this case we will have the choice on which lone variable we wish to solve for, either will give the same final result.

Example 9.

$\boldsymbol{x} + y = 5$ Find the lone variable: x or y in first, or x in second.
$x - y = -1$ We will chose x in the first

$\boldsymbol{x} + y = 5$ Solve for the lone variable, subtract y from both sides

$\underline{\boldsymbol{-y \ -y}}$

$x = 5 - y$ Plug into the untouched equation, the second equation

$(\boldsymbol{5 - y}) - y = -1$ Solve, parenthesis are not needed here, combine like terms

$5 - 2y = -1$ Subtract 5 from both sides

$\underline{-5 \qquad -5}$

$-2y = -6$ Divide both sides by -2

$\overline{-2} \quad \overline{-2}$

$y = 3$ We have our y!

$x = 5 - (3)$ Plug into lone variable equation, evaluate

$x = 2$ Now we have our x

$$(2,3) \quad \text{Our Solution}$$

Just as with graphing it is possible to have no solution $\varnothing$ (parallel lines) or infinite solutions (same line) with the substitution method. While we won't have a parallel line or the same line to look at and conclude if it is one or the other, the process takes an interesting turn as shown in the following example.

Example 10.

$$\begin{array}{rl}
\begin{array}{r}\boldsymbol{y}+4=3x\\ 2y-6x=-8\end{array} & \text{Find the lone varaible, } y \text{ in the first equation}\\
y+4=3x & \text{Solve for the lone variable, subtract 4 from both sides}\\
\underline{-4\quad -4} & \\
y=3x-4 & \text{Plug into untouched equation}\\
2(\boldsymbol{3x-4})-6x=-8 & \text{Solve, distribute through parenthesis}\\
6x-8-6x=-8 & \text{Combine like terms } 6x-6x\\
-8=-8 & \text{Variables are gone! } A \text{ true statement.}\\
\text{Infinite solutions} & \text{Our Solution}
\end{array}$$

Because we had a true statement, and no variables, we know that anything that works in the first equation, will also work in the second equation. However, we do not always end up with a true statement.

Example 11.

$$\begin{array}{rl}
\begin{array}{r}6x-3y=-9\\ -2x+\boldsymbol{y}=5\end{array} & \text{Find the lone variable, } y \text{ in the second equation}\\
-2x+y=5 & \text{Solve for the lone variable, add } 2x \text{ to both sides}\\
\underline{+2x\quad +2x} & \\
y=5+2x & \text{Plug into untouched equation}\\
6x-3(\boldsymbol{5+2x})=-9 & \text{Solve, distribute through parenthesis}\\
6x-15-6x=-9 & \text{Combinel ike terms } 6x-6x\\
-15\neq-9 & \text{Variables are gone! } A \text{ false statement.}\\
\text{No Solution } \varnothing & \text{Our Solution}
\end{array}$$

Because we had a false statement, and no variables, we know that nothing will work in both equations.

One more question needs to be considered, what if there is no lone variable? If there is no lone variable substitution can still work to solve, we will just have to select one variable to solve for and use fractions as we solve.

Example 12.

$$
\begin{array}{rl}
5\boldsymbol{x} - 6y = -14 & \text{No lone variable,} \\
-2x + 4y = 12 & \text{we will solve for } x \text{ in the first equation} \\
5x - 6y = -14 & \text{Solve for our variable, add } 6y \text{ to both sides} \\
\underline{\mathbf{+6y \;\; +6y}} & \\
5x = -14 + 6y & \text{Divide each term by 5} \\
\overline{\mathbf{5}} \quad \overline{\mathbf{5}} \quad \overline{\mathbf{5}} & \\
x = \frac{-14}{5} + \frac{6y}{5} & \text{Plug into untouched equation} \\
-2\left(\frac{-14}{5} + \frac{6y}{5}\right) + 4y = 12 & \text{Solve, distribute through parenthesis} \\
\frac{28}{5} - \frac{6y}{5} + 4y = 12 & \text{Clear fractions by multiplying by 5} \\
\frac{28(\mathbf{5})}{5} - \frac{12y(\mathbf{5})}{5} + 4y(\mathbf{5}) = 12(\mathbf{5}) & \text{Reduce fractions and multiply} \\
28 - 12y + 20y = 60 & \text{Combine like terms } -12y + 20y \\
28 + 8y = 60 & \text{Subtract 28 from both sides} \\
\underline{\mathbf{-28 \qquad -28}} & \\
8y = 32 & \text{Divide both sides by 8} \\
\overline{\mathbf{8}} \quad \overline{\mathbf{8}} & \\
y = 4 & \text{We have our } y \\
x = \frac{-14}{5} + \frac{6(4)}{5} & \text{Plug into lone variable equation, multiply} \\
x = \frac{-14}{5} + \frac{24}{5} & \text{Add fractions} \\
x = \frac{10}{5} & \text{Reduce fraction} \\
x = 2 & \text{Now we have our } x \\
(2, 4) & \text{Our Solution}
\end{array}
$$

Using the fractions does make the problem a bit more tricky. This is why we have another method for solving systems of equations that will be disucessed in another lesson.

Practice - Solving with Substitution

Solve each system by substitution.

1) $y = -3x$
$y = 6x - 9$

3) $y = -2x - 9$
$y = 2x - 1$

5) $y = 6x + 4$
$y = -3x - 5$

7) $y = 3x + 2$
$y = -3x + 8$

9) $y = 2x - 3$
$y = -2x + 9$

11) $y = 6x - 6$
$-3x - 3y = -24$

13) $y = -6$
$3x - 6y = 20$

15) $y = -5$
$3x + 4y = -17$

17) $-2x + 2y = 18$
$y = 7x + 15$

19) $y = -8x + 19$
$-x + 6y = 16$

21) $7x - 2y = -7$
$y = 7$

23) $x - 5y = 7$
$2x + 7y = -20$

25) $-2x - y = -10$
$x - 8y = -23$

27) $-6x + y = 20$
$-3x - 3y = -18$

29) $3x + y = 9$
$2x + 8y = -16$

31) $2x + y = 2$
$3x + 7y = 14$

33) $x + 5y = 15$
$-3x + 7y = 6$

35) $-2x + 4y = -16$
$y = -2$

37) $-6x + 6y = -12$
$8x - 3y = 16$

39) $2x + 3y = 16$
$-7x - y = 20$

2) $y = x + 5$
$y = -2x - 4$

4) $y = -6x + 3$
$y = 6x + 3$

6) $y = 3x + 13$
$y = -2x - 22$

8) $y = -2x - 9$
$y = -5x - 21$

10) $y = 7x - 24$
$y = -3x + 16$

12) $-x + 3y = 12$
$y = 6x + 21$

14) $6x - 4y = -8$
$y = -6x + 2$

16) $7x + 2y = -7$
$y = 5x + 5$

18) $y = x + 4$
$3x - 4y = -19$

20) $y = -2x + 8$
$-7x - 6y = -8$

22) $x - 2y = -13$
$4x + 2y = 18$

24) $3x - 4y = 15$
$7x + y = 4$

26) $6x + 4y = 16$
$-2x + y = -3$

28) $7x + 5y = -13$
$x - 4y = -16$

30) $-5x - 5y = -20$
$-2x + y = 7$

32) $2x + y = -7$
$5x + 3y = -21$

34) $2x + 3y = -10$
$7x + y = 3$

36) $-2x + 2y = -22$
$-5x - 7y = -19$

38) $-8x + 2y = -6$
$-2x + 3y = 11$

40) $-x - 4y = -14$
$-6x + 8y = 12$

4.3

Systems of Equations - Addition/Elimination

When solving systems we have found that graphing is very limited when solving equations. We then considered a second method known as substituion. This is probably the most used idea in solving systems in various areas of algebra. However, substitution can get ugly if we don't have a lone variable. This leads us to our second method for solving systems of equations. This method is known as either Elimination or Addition. We will set up the process in the following examples, then define the five step process we can use to solve by elimination.

Example 13.

$$\begin{array}{rl}
\begin{array}{r} 3x-4y=8 \\ \underline{5x+4y=-24} \end{array} & \text{Notice opposites in front of } y's\text{. Add collumns.} \\
\frac{8x}{8} \quad = \frac{-16}{8} & \text{Solve for } x\text{, divide by } 8 \\
x=-2 & \text{We have our } x! \\
5(-2)+4y=-24 & \text{Plug into either origional equation, simplify} \\
-10+4y=-24 & \text{Add } 10 \text{ to both sides} \\
\underline{+10 \qquad +10} & \\
\frac{4y}{4}=\frac{-14}{4} & \text{Divide by } 4 \\
y=\frac{-7}{2} & \text{Now we have our } y! \\
\left(-2,\frac{-7}{2}\right) & \text{Our Solution}
\end{array}$$

In the previous example one variable had opposites in front of it, $-4y$ and $4y$. Adding these together eliminated the y completely. This allowed us to solve for the x. This is the idea behind the addition method. However, generally we won't have opposites in front of one of the variables. In this case we will manipulate the equations to get the opposites we want by multiplying one or both equations (on both sides!). This is shown in the next example.

Example 14.

$$\begin{array}{rl}
-6x+5y=22 & \text{We can get opposites in front of } x\text{, by multiplying the} \\
2x+3y=2 & \text{second equation by } 3\text{, to get } -6x \text{ and } +6x
\end{array}$$

$$\begin{array}{rl}
3(2x+3y)=(2)3 & \text{Distribute to get new second equation.} \\
6x+9y=6 & \text{New second equation}
\end{array}$$

$$
\begin{array}{rl}
\underline{-6x+5y=22} & \text{First equation still the same, add} \\
14y=28 & \text{Divide both sides by 14} \\
\overline{14}\quad\overline{14} & \\
y=2 & \text{We have our } y! \\
2x+3(2)=2 & \text{Plug into one of the origional equations, simplify} \\
2x+6=2 & \text{Subtract 6 from both sides} \\
\underline{-6-6} & \\
2x=-4 & \text{Divide both sides by 2} \\
\overline{2}\quad\overline{2} & \\
x=-2 & \text{We also have our } x! \\
(-2,2) & \text{Our Solution}
\end{array}
$$

When we looked at the x terms, $-6x$ and $2x$ we decided to multiply the $2x$ by 3 to get the opposites we were looking for. What we are looking for with our opposites is the least common multiple (LCM) of the coefficients. We also could have solved the above problem by looking at the terms with y, $5y$ and $3y$. The LCM of 3 and 5 is 15. So we would want to multiply both equations, the $5y$ by 3, and the $3y$ by -5 to get opposites, $15y$ and $-15y$. This illustrates an important point, some problems we will have to multiply both equations by a constant (on both sides) to get the opposites we want.

Example 15.

$$
\begin{array}{ll}
3x+6y=-9 & \text{We can get opposites in front of } x\text{, find LCM of 6 and 9,} \\
2x+9y=-26 & \text{The LCM is 18. We will multiply to get } 18y \text{ and } -18y
\end{array}
$$

$$
\begin{array}{rl}
3(3x+6y)=(-9)3 & \text{Multiply the first equation by 3, both sides!} \\
9x+18y=-27 &
\end{array}
$$

$$
\begin{array}{rl}
-2(2x+9y)=(-26)(-2) & \text{Multiply the second equation by } -2\text{, both sides!} \\
-4x-18y=52 &
\end{array}
$$

$$
\begin{array}{rl}
9x+18y=-27 & \text{Add two new equations together} \\
\underline{-4x-18y=52} & \\
5x \qquad =25 & \text{Divide both sides by 5} \\
\overline{5} \qquad\quad \overline{5} & \\
x=5 & \text{We have our solution for } x \\
3(5)+6y=-9 & \text{Plug into either origional equation, simplify} \\
15+6y=-9 & \text{Subtract 15 from both sides} \\
\underline{-15 \qquad -15} &
\end{array}
$$

$$\begin{array}{rl} \frac{6y}{6} = \frac{-24}{6} & \text{Divide both sides by } 6 \\ y = -4 & \text{Now we have our solution for } y \\ (5, -4) & \text{Our Solution} \end{array}$$

It is important for each problem as we get started that all variables and constants are lined up before we start multiplying and adding equations. This is illustrated in the next example which includes the five steps we will go through to solve a problem using elimination.

Problem	$2x - 5y = -13$ $-3y + 4 = -5x$
1. Line up the variables and constants	Second Equation: $-3y + 4 = -5x$ $+5x - 4 \quad +5x - 4$ $5x - 3y = -4$
2. Multiply to get opposites (use LCD)	$2x - 5y = -13$ $5x - 3y = -4$ First Equation: multiply by -5 $-5(2x - 5y) = (-13)(-5)$ $-10x + 25y = 65$ Second Equation: multiply by 2 $2(5x - 3y) = (-4)2$ $10x - 6y = -8$ $-10x + 25y = 65$ $10x - 6y = -8$
3. Add	$19y = 57$
4. Solve	$\frac{19y}{19} = \frac{57}{19}$ $y = 3$
5. Plug into either origional and solve	$2x - 5(3) = -13$ $2x - 15 = -13$ $+15 \quad +15$ $\frac{2x}{2} = \frac{2}{2}$ $x = 1$
Solution	$(1, 3)$

Just as with graphing and substution, it is possible to have no solution or infinite solutions with elimination. Just as with substitution, if the variables all dissapear from our problem, a true statment will indicate infinite solutions and a false statment will indicate no solution.

Example 16.

$$\begin{array}{rl} 2x-5y=3 & \\ -6x+15y=-9 & \text{To get opposites in front of } x\text{, multiply first equation by 3} \end{array}$$

$$\begin{array}{rl} 3(2x-5y)=(3)3 & \text{Distribute} \\ 6x-15y=9 & \end{array}$$

$$\begin{array}{rl} 6x-15y=9 & \text{Add equations together} \\ \underline{-6x+15y=-9} & \\ 0=0 & \text{True statment} \\ \text{Infinite solutions} & \text{Our Solution} \end{array}$$

Example 17.

$$\begin{array}{rl} 4x-6y=8 & \\ 6x-9y=15 & \text{LCM for } x's \text{ is 12.} \end{array}$$

$$\begin{array}{rl} 3(4x-6y)=(8)3 & \text{Multiply first equation by 3} \\ 12x-18y=24 & \end{array}$$

$$\begin{array}{rl} -2(6x-9y)=(15)(-2) & \text{Multiply second equation by } -2 \\ -12x+18y=-30 & \end{array}$$

$$\begin{array}{rl} 12x-18y=24 & \text{Add both new equations together} \\ \underline{-12x+18y=-30} & \\ 0=-6 & \text{False statment} \\ \text{No Solution} & \text{Our Solution} \end{array}$$

We have covered three different methods that can be used to solve a system of two equations with two variables. While all three can be used to solve any system, graphing works great for small integer solutions. Substitution works great when we have a lone variable, and addition works great when the other two methods fail. As each method has its own strengths, it is important you are familiar with all three methods.

Practice - Solving with Addition/Elimination

Solve each system by elimination.

1) $4x+2y=0$
$-4x-9y=-28$

2) $-7x+y=-10$
$-9x-y=-22$

3) $-9x+5y=-22$
$9x-5y=13$

4) $-x-2y=-7$
$x+2y=7$

5) $-6x+9y=3$
$6x-9y=-9$

6) $5x-5y=-15$
$5x-5y=-15$

7) $4x-6y=-10$
$4x-6y=-14$

8) $-3x+3y=-12$
$-3x+9y=-24$

9) $-x-5y=28$
$-x+4y=-17$

10) $-10x-5y=0$
$-10x-10y=-30$

11) $2x-y=5$
$5x+2y=-28$

12) $-5x+6y=-17$
$x-2y=5$

13) $10x+6y=24$
$-6x+y=4$

14) $x+3y=-1$
$10x+6y=-10$

15) $2x+4y=-4$
$4x-12y=8$

16) $-6x+4y=12$
$12x+6y=18$

17) $-7x+4y=-4$
$10x-8y=-8$

18) $-6x+4y=4$
$-3x-y=26$

19) $5x+10y=20$
$-6x-5y=-3$

20) $-9x-5y=-19$
$3x-7y=-11$

21) $-7x-3y=12$
$-6x-5y=20$

22) $-5x+4y=4$
$-7x-10y=-10$

23) $9x-2y=-18$
$5x-7y=-10$

24) $3x+7y=-8$
$4x+6y=-4$

25) $9x+6y=-21$
$-10x-9y=28$

26) $-4x-5y=12$
$-10x+6y=30$

27) $-7x+5y=-8$
$-3x-3y=12$

28) $8x+7y=-24$
$6x+3y=-18$

29) $-8x-8y=-8$
$10x+9y=1$

30) $-7x+10y=13$
$4x+9y=22$

31) $9y=7-x$
$-18y+4x=-26$

32) $0=-9x-21+12y$
$1+\frac{4}{3}y+\frac{7}{3}x=0$

33) $0=9x+5y$
$y=\frac{2}{7}x$

34) $-6-42y=-12x$
$x-\frac{1}{2}-\frac{7}{2}y=0$

4.4

Systems of Equations - 3 Variables

Solving systems of equations with 3 variables is very similar to how we solve systems with two varaibles. When we had two variables we reduced the system down to one with only one variable (by substitution or addition). With three variables we will reduce the system down to one with two variables (usually by addition), which we can then solve by either addition or substitution.

To reduce from three variables down to two it is very important to keep the work ogranized. We will use addition with two equations to eliminate one variable. This new equation we will call (A). Then we will use a different pair of equations and use addition to eliminate the **same** variable. This second new equation we will call (B). Once we have done this we will have two equations (A) and (B) with the same two variables that we can solve using either method. This is shown in the following examples.

Example 18.

$$\begin{aligned} 3x+2y-z&=-1 \\ -2x-2y+3z&=5 \\ 5x+2y-z&=3 \end{aligned}$$ We will eliminate y using two different pairs of equations

$$\begin{array}{lr} & 3x+2y-z=-1 \\ & \underline{-2x-2y+3z=5} \\ (A) & x \qquad +2z=4 \end{array}$$

Using the first two equations,
Add the first two equations
This is equation (A), our first equation

$$\begin{array}{lr} & -2x-2y+3z=5 \\ & \underline{5x+2y-z=3} \\ (B) & 3x \qquad +2z=8 \end{array}$$

Using the second two equations
Add the second two equations
This is equation (B), our second equation

$$\begin{aligned} (A)\ \ x+2z&=4 \\ (B)\ \ 3x+2z&=8 \end{aligned}$$

Using (A) and (B) we will solve this system.
We will solve by addition

$$\begin{aligned} -1(x+2z)&=(4)(-1) \\ -x-2z&=-4 \end{aligned}$$

Multiply (A) by -1

$$\begin{aligned} -x-2z&=-4 \\ \underline{3x+2z}&\underline{=8} \\ 2x&=4 \\ \overline{2}\ \ &\ \ \overline{2} \\ x&=2 \end{aligned}$$

Add to the second equation, unchanged
Solve, divide by 2
We now have x! Plug this into either (A) or (B)

$$\begin{aligned} (2)+2z&=4 \\ \underline{-2\qquad}&\underline{\ \ -2} \\ 2z&=2 \\ \overline{2}\ \ &\ \ \overline{2} \\ z&=1 \end{aligned}$$

We plug it into (A), solve this equation, subtract 2
Divide by 2
We now have z! Plug this and x into any origional equation

$$\begin{aligned} 3(2)+2y-(1)&=-1 \\ 2y+5&=-1 \\ \underline{-5}\ \ &\underline{\ \ -5} \end{aligned}$$

We use the first, multiply $3(2)=6$ and combine with -1
Solve, subtract 5

$$\frac{2y}{2} = \frac{-6}{2} \quad \text{Divide by 2}$$

$$y = -3 \quad \text{We now have } y!$$

$$(2, -3, 1) \quad \text{Our Solution}$$

As we are solving for x, y, and z we will have an ordered triplet (x, y, z) instead of just the ordered pair (x, y). In this above problem, y was easily eliminated using the addition method. However, sometimes we may have to do a bit of work to get a variable to eliminate. Just as with addition of two equations, we may have to multiply equations by something on both sides to get the opposites we want so a variable eliminates. As we do this remmeber it is improtant to eliminate the **same** variable both times using two **different** pairs of equations.

Example 19.

$$\begin{aligned} 4x - 3y + 2z &= -29 && \text{No variable will easily eliminate.} \\ 6x + 2y - z &= -16 && \text{We could choose any variable, so we chose } x \\ -8x - y + 3z &= 23 && \text{We will eliminate } x \text{ twice.} \end{aligned}$$

$$\begin{aligned} 4x - 3y + 2z &= -29 && \text{Start with first two equations. LCM of 4 and 6 is 12.} \\ 6x + 2y - z &= -16 && \text{Make the first equation have } 12x\text{, the second } -12x \end{aligned}$$

$$\begin{aligned} 3(4x - 3y + 2z) &= (-29)3 && \text{Multiply the first equation by 3} \\ 12x - 9y + 6z &= -87 \end{aligned}$$

$$\begin{aligned} -2(6x + 2y - z) &= (-16)(-2) && \text{Multiply the second equation by } -2 \\ -12x - 4y + 2z &= 32 \end{aligned}$$

$$\begin{aligned} 12x - 9y + 6z &= -87 && \text{Add these two equations together} \\ \underline{-12x - 4y + 2z} &\underline{= 32} \\ (A) \qquad -13y + 8z &= -55 && \text{This is our } (A) \text{ equation} \end{aligned}$$

$$\begin{aligned} 6x + 2y - z &= -16 && \text{Now use the second two equations (}a\text{ different pair)} \\ -8x - y + 3z &= 23 && \text{The LCM of 6 and } -8 \text{ is 24.} \end{aligned}$$

$$\begin{aligned} 4(6x + 2y - z) &= (-16)4 && \text{Multiply the first equation by 4} \\ 24x + 8y - 4 &= -64 \end{aligned}$$

$$
\begin{array}{rl}
3(-8x-y+3z)=(23)3 & \text{Multiply the second equation by 3}\\
-24x-3y+9z=69 & \\
\\
24x+8y-4=-64 & \text{Add these two equations together}\\
\underline{-24x-3y+9z=69} & \\
(B)\qquad\qquad 5y+5z=5 & \text{This is our } (B) \text{ equation}\\
\\
(A)\quad -13y+8z=-55 & \text{Using } (A) \text{ and } (B) \text{ we will solve this system}\\
(B)\qquad 5y+5z=5\quad & \text{The second equation is solved for } z \text{ to use substitution}\\
\\
5y+5z=5 & \text{Solving for } z\text{, subtract } 5y\\
\underline{-5y\qquad -5y} & \\
5z=5-5y & \text{Divide each term by 5}\\
\overline{5}\quad\overline{5}\quad\overline{5} & \\
z=1-y & \text{Plug into untouched equation}\\
-13y+8(1-y)=-55 & \text{Distribute}\\
-13y+8-8y=-55 & \text{Combine like terms } -13y-8y\\
-21y+8=-55 & \text{Subtract 8}\\
\underline{-8\qquad -8} & \\
-21y=-63 & \text{Divide by } -21\\
\overline{-21}\quad\overline{-21} & \\
y=3 & \text{We have our } y\text{! Plug this into } z= \text{ equations}\\
z=1-(3) & \text{Evaluate}\\
z=-2 & \text{We have } z\text{, now find } x \text{ from original equation.}\\
\\
4x-3(3)+2(-2)=-29 & \text{Multiply and combine like terms}\\
4x-13=-29 & \text{Add 13}\\
\underline{+13\quad +13} & \\
4x=-16 & \text{Divide by 4}\\
\overline{4}\qquad\overline{4} & \\
x=-4 & \text{We have our } x\text{!}\\
\\
(-4,3,-2) & \text{Our Solution!}
\end{array}
$$

Just as with two variables and two equations, we can have special cases come up with three variables and three equations. The way we interpret the result is identical.

Example 20.

$$\begin{aligned}
5x - 4y + 3z &= -4 \\
-10x + 8y - 6z &= 8 && \text{We will eliminate } x\text{, start with first two equations} \\
15x - 12y + 9z &= -12
\end{aligned}$$

$$\begin{aligned}
5x - 4y + 3z &= -4 && \text{LCM of 5 and } -10 \text{ is 10.} \\
-10x + 8y - 6z &= 8
\end{aligned}$$

$$\begin{aligned}
2(5x - 4y + 3z) &= -4(2) && \text{Multiply the first equation by 2} \\
10x - 8y + 6z &= -8
\end{aligned}$$

$$\begin{aligned}
10x - 8y + 6z &= -8 && \text{Add this to the second equation, unchanged} \\
\underline{-10x + 8y - 6 = 8} & \\
0 &= 0 && A \text{ true statment} \\
\text{Infinite Solutions} & && \text{Our Solution}
\end{aligned}$$

Example 21.

$$\begin{aligned}
3x - 4y + z &= 2 \\
-9x + 12y - 3z &= -5 && \text{We will eliminate } z\text{, starting with the first two equations} \\
4x - 2y - z &= 3
\end{aligned}$$

$$\begin{aligned}
3x - 4y + z &= 2 && \text{The LCM of 1 and } -3 \text{ is 3} \\
-9x + 12y - 3z &= -5
\end{aligned}$$

$$\begin{aligned}
3(3x - 4y + z) &= (2)3 && \text{Multiply the first equation by 3} \\
9x - 12y + 3z &= 6
\end{aligned}$$

$$\begin{aligned}
9x - 12y + 3z &= 6 && \text{Add this to the second equation, unchanged} \\
\underline{-9x + 12y - 3z = -5} & \\
0 &= 1 && A \text{ false statement} \\
\text{No Solution } \varnothing & && \text{Our Solution}
\end{aligned}$$

Equations with three (or more) variables are not any more difficult than two variables if we are careful to keep our information organized and eliminate the same variable twice using two different pairs of equations. It is possible to solve each system several different ways. We can use different pairs of equations or eliminate variables in different orders, but as long as our information is organized and our algebra is correct, we will arrive at the same final solution.

Practice - Solving Equations with 3 Variables

Solve each of the following systems of equation.

1) $a-2b+c=5$
$2a+b-c=-1$
$3a+3b-2c=-4$

3) $3x+y-z=11$
$x+3y=z+13$
$x+y-3z=11$

5) $x+6y+3z=4$
$2x+y+2z=3$
$3x-2y+z=0$

7) $x+y+z=6$
$2x-y-z=-3$
$x-2y+3z=6$

9) $x+y-z=0$
$x-y-z=0$
$x+y+2z=0$

11) $-2x+y-3z=1$
$x-4y+z=6$
$4x+16y+4z=24$

13) $2x+y-3z=0$
$x-4y+z=0$
$4x+16y+4z=0$

15) $3x+2y+2z=3$
$x+2y-z=5$
$2x-4y+z=0$

17) $x-2y+3z=4$
$2x-y+z=-1$
$4x+y+z=1$

19) $x-y+2z=0$
$x-2y+3z=-1$
$2x-2y+z=-3$

21) $4x-3y+2z=40$
$5x+9y-7z=47$
$9x+8y-3z=97$

23) $3x+3y-2z=13$
$6x+2y-5z=13$
$5x-2y-5z=1$

25) $3x-4y+2z=1$
$2x+3y-3z=-1$
$x+1\text{oy}-8z=7$

27) $m+6n+3p=8$
$3m+4n=-3$
$5m+7n=1$

29) $-2w+2x+2y-2z=-10$
$w+x+y+z=-5$
$3w+2x+2y+4z=1$
$w+3x-2y+2z=-6$

31) $w+x+y+z=2$
$w+2x+2y+4z=1$
$-w+x-y-z=-6$
$-w+3x+y-z=-2$

2) $2x+3y=z-1$
$3x=8z-1$
$5y+7z=-1$

4) $x+y+z=2$
$6x-4y+5z=31$
$5x+2y+2z=13$

6) $x-y+2z=-3$
$x+2y+3z=4$
$2x+y+z=-3$

8) $x+y-z=0$
$x+2y-4z=0$
$2x+y+z=0$

10) $x+2y-z=4$
$4x-3y+z=8$
$5x-y=12$

12) $4x+12y+16z=4$
$3x+4y+5z=3$

$x+8y+11z=1$

14) $4x+12y+16z=0$
$3x+4y+5z=0$
$x+8y+11z=0$

16) $p+q+r=1$
$p+2q+3r=4$
$4p+5q+6r=7$

18) $x+2y-3z=9$
$2x-y+2z=-8$
$3x-y-4z=3$

20) $4x-7y+3z=1$
$3x+y-2z=4$
$4x-7y+3z=6$

22) $3x+y-z=10$
$8x-y-6z=-3$
$5x-2y-5z=1$

24) $2x-3y+5z=1$
$3x+2y-z=4$
$4x+7y-7z=7$

26) $2x+y=z$
$4x+z=4y$
$y=x+1$

28) $3x+2y=z+2$
$y=1-2x$
$3z=-2y$

30) $-w+2x-3y+z=-8$
$-w+x+y-z=-4$
$w+x+y+z=22$
$-w+x-y-z=-4$

32) $w+x-y+z=0$
$-w+2x+2y+z=5$
$-w+3x+y-z=-4$
$-2w+x+y-3z=-7$

4.5

Systems of Equations - Value Problems

One application of system of equations are known as value problems. Value problems are ones where each variable has a value attached to it. For example, if our variable is the number of nickles in a person's pocket, those nickles would have a value of five cents each. We will use a table to help us set up and solve value problems. The basic structure of the table is shown below.

	Number	Value	Total
Item 1			
Item 2			
Total			

The first column in the table is used for the number of things we have. Quite often, this will be our variables. The second column is used for the the value each item has. The third column is used for the total value which we calculate by multiplying the number by the value. For example, if we have 7 dimes, each with a value of 10 cents, the total value is $7 \cdot 10 = 70$ cents. The last row of the table is for totals. We only will use the third row (also marked total) for the totals that are given to use. This means sometimes this row may have some blanks in it. Once the table is filled in we can easily make equations by adding each column, setting it equal to the total at the bottom of the column. This is shown in the following example.

Example 22.

In a child's bank are 11 coins that have a value of $1.85. The coins are either quarters or dimes. Howe many each does the child have?

	Number	Value	Total
Quarter	q	25	
Dime	d	10	
Total			

Using value table, use q for quarters, d for dimes
Each quarter$'s$ value is 25 cents, dime$'s$ is 10 cents

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total			

Multiply number by value to get totals

	Number	Value	Total
Quarter	q	25	$25q$
Dime	d	10	$10d$
Total	11		185

We have 11 coins total. This is the number total.
We have 1.85 for the final total,
Write final total in cents (185)
Because 25 and 10 are cents

$$q + d = 11$$ First and last columns are our equations by adding

$$25q + 10d = 185$$ Solve by either addition or substitution.

$$-10(q + d) = (11)(-10)$$ Using addition, multiply first equation by -10

$$-10q - 10d = -110$$

$$-10q - 10d = -110$$ Add together equations

$$\underline{25q + 10d = 185}$$

$$\frac{15q}{15} = \frac{75}{15}$$ Divide both sides by 15

$$q = 5$$ We have our q, number of quarters is 5

$$(5) + d = 11$$ Plug into one of origional equations

$$\underline{-5 \qquad -5}$$ Subtract 5 from both sides

$$d = 6$$ We have our d, number of dimes is 6

5 quarters and 6 dimes Our Solution

World View Note: American coins are the only coins that do not state the value of the coin. On the back of the dime it says “one dime” (not 10 cents). On the back of the quarter it says “one quarter” (not 25 cents). On the penney it says “one cent” (not 1 cent). The rest of the world (Euros, Yen, Pesos, etc) all write the value as a number so people who don’t speak the language can easily use the coins.

Ticket sales also have a value. Often different types of tickets sell for different prices (values). These problems can be solve in much the same way.

Example 23.

There were 41 tickets sold for an event. Tickets for children cost \$1.50 and tickets for adults cost \$2.00. Total receipts for the event were \$73.50. How many of each type of ticket were sold?

	Number	Value	Total
Child	c	1.5	
Adult	a	2	
Total			

Using our value table, c for child, a for adult
Child tickets have value 1.50, adult value is 2.00
(we can drop the zeros after the decimal point)

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total			

Multiply number by value to get totals

	Number	Value	Total
Child	c	1.5	$1.5c$
Adult	a	2	$2a$
Total	41		73.5

We have 41 tickets sold. This is our number total
The final total was 73.50
Write in dollars as 1.5 and 2 are also dollars

$$c + a = 41$$ First and last columns are our equations by adding

$$1.5c + 2a = 73.5$$ We can solve by either addition or substitution

$$c + a = 41$$ We will solve by substitution.

$$\underline{-c \qquad -c}$$ Solve for a by subtracting c

$$a = 41 - c$$

$$1.5c + 2(41 - c) = 73.5$$ Substitute into untouched equation

$$1.5c + 82 - 2c = 73.5$$ Distribute

$$-0.5c + 82 = 73.5$$ Combine like terms

$$\underline{-82 - 82}$$ Subtract 82 from both sides

$$-0.5c = -8.5$$ Divide both sides by -0.5

$$\overline{-0.5} \quad \overline{-0.5}$$

$c=17$	We have c, number of child tickets is 17
$a=41-(17)$	Plug into $a=$ equation to find a
$a=24$	We have our a, number of adult tickets is 24
17 child tickets and 24 adult tickets	Our Solution

Some problems will not give us the total number of items we have. Instead they will give a relationship between the items. Here we will have statments such as "There are twice as many dimes as nickles". While it is clear that we need to multiply one variable by 2, it may not be clear which variable gets multiplied by 2. Generally the equations are backwards from the english sentence. If there are twice as many dimes, than we multiply the other variable (nickels) by two. So the equaion would be $d=2n$. This type of problem is in the next example.

Example 24.

A man has a collection of stamps made up of 5 cent stamps and 8 cent stamps. There are three times as many 8 cent stamps as 5 cent stamps. The total value of all the stamps is $3.48. How many of each stamp does he have?

	Number	Value	Total
Five	f	5	
Eight	e	8	
Total			

Use value table, f for five cent stamp, and e for eight
Also list value of each stamp under value column

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			

Multiply number by value to get total

	Number	Value	Total
Five	f	5	$5f$
Eight	e	8	$8e$
Total			348

The final total was 338 (written in cents)
We do not know the total number, this is left blank.

$e=3f$	**Three times as many eight cent stampls as five cent stamps**
$5f+8e=348$	Total column gives second equation
$5f+8(3f)=348$	Substitution, substitute first equation in second
$5f+24f=348$	Multiply first
$29f=348$	Combine like terms
$\overline{29} \quad \overline{29}$	Divide both sides by 39
$f=12$	We have f. There are 12 five cent stamps
$e=3(12)$	Plug into first equation

$$e = 36 \quad \text{We have } e\text{, There are 36 eight cent stamps}$$

12 five cent, 36 eight cent stamps — Our Solution

The same process for solving value problems can be applied to solving interest problems. Our table titles will be adjusted slightly as we do so.

	Invest	Rate	Interest
Account 1			
Account 2			
Total			

Our first column is for the amount invested in each account. The second column is the interest rate earned (written as a decimal - move decimal point twice left), and the last column is for the amount of interset earned. Just as before, we multiply the investment amount by the rate to find the final column, the interst earned. This is shown in the following example.

Example 25.

A woman invests \$4000 in two accounts, one at 6% inteset, the other at 9% interest for one year. At the end of the year she had earned \$270 in interest. How much did she have invested in each account?

	Invest	Rate	Interest
Account 1	x	0.06	
Account 2	y	0.09	
Total			

Use our investment table, x and y for accounts
Fill in interest rates as decimals

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total			

Multiply across to find interest earned.

	Invest	Rate	Interest
Account 1	x	0.06	$0.06x$
Account 2	y	0.09	$0.09y$
Total	4000		270

Total investment is 4000,
Total interest was 276

$$x + y = 4000 \quad \text{First and last column give our two equations}$$
$$0.06x + 0.09y = 270 \quad \text{Solve by either substitution or addition}$$

$$-0.06(x + y) = (4000)(-0.06) \quad \text{Use Addition, multiply first equation by } -0.06$$
$$-0.06x - 0.06y = -240$$

$$
\begin{array}{rl}
-0.06x-0.06y=-240 & \text{Add equations together}\\
\underline{0.06x+0.09y=270} & \\
0.03y=30 & \text{Divide both sides by } 0.03\\
\overline{0.03}\quad\overline{0.03} & \\
y=1000 & \text{We have } y, \$1000 \text{ invested at } 9\%\\
x+1000=4000 & \text{Plug into origional equation}\\
-1000\;-1000 & \text{Subtract } 1000 \text{ from both sides}\\
x=3000 & \text{We have } x, \$3000 \text{ invested at } 6\%\\
\$1000 \text{ at } 9\% \text{ and } \$3000 \text{ at } 6\% & \text{Our Solution}
\end{array}
$$

The same process can be used to find an unknown interest rate.

Example 26.

John invests \$5000 in one account and \$8000 in an account paying 4% more in interest. He earned \$1230 in interest after one year. At what rates did he invest?

	Invest	Rate	Interest
Account 1	5000	x	
Account 2	8000	$x+0.04$	
Total			

Our investment table. Use x for first rate
The second rate is 4% higher, or $x+0.04$
Be sure to write this rate as *a* decimal!

	Invest	Rate	Interest
Account 1	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000x+320$
Total			

Multiply to fill in interest column.
Be sure to distribute $8000(x+0.04)$

	Invest	Rate	Interest
Account 2	5000	x	$5000x$
Account 2	8000	$x+0.04$	$8000x+320$
Total			1230

Total interest was 1230.

$$
\begin{array}{rl}
5000x+8000x+320=1230 & \text{Last column gives our equation}\\
13000x+320=1230 & \text{Combine like terms}\\
\underline{-320\;\;-320} & \text{Subtract } 320 \text{ from both sides}\\
13000x=910 & \text{Divide both sides by } 13000\\
\overline{13000}\;\overline{13000} & \\
x=0.07 & \text{We have our } x, 7\% \text{ interst}\\
(0.07)+0.04 & \text{Second account is } 4\% \text{ higher}\\
0.11 & \text{The account with } \$8000 \text{ is at } 11\%\\
\$5000 \text{ at } 7\% \text{ and } \$8000 \text{ at } 11\% & \text{Our Solution}
\end{array}
$$

Practice - Value Problems

Solve.

1) A collection of dimes and quaters is worth \$15.25. There are 103 coins in all. How many of each is there?

2) A collection of half dollars and nickels is worth \$13.40. There are 34 coins in all. How many are there?

3) The attendance at a school concert was 578. Admission was \$2.00 for adults and \$1.50 for children. The total receipts were \$985.00. How many adults and how many children attended?

4) A purse contains \$3.90 made up of dimes and quarters. If there are 21 coins in all, how many dimes and how many quarters were there?

5) A boy has \$2.25 in nickels and dimes. If there are twice as many dimes as nickels, how many of each kind has he?

6) \$3.75 is made up of quarters and half dollars. If the number of quarters exceeds the number of half dollars by 3, how many coins of each denomination are there?

7) A collection of 27 coins consisitng of nickels and dimes amounts to \$2.25. How many coins of each kind are there?

8) \$3.25 in dimes and nickels, were distributed amoung 45 boys. If each received one coin, how many received dimes and how many received nickels?

9) There were 429 people at a play. Admission was \$1 each for adults and 75 cents each for children. The receipts were \$372.50. How many children and how many adults attended?

10) There were 200 tickets sold for a women's basketball game. Tickets for students were 50 cents each anf for adults 75 cents each. The total amount of money collected was \$132.50. How many of each type of ticket was sold?

11) There were 203 tickets sold for a volleyball game. For activity-card holders, the price was \$1.25 each and for noncard holders the price was \$2 each. The total amount of money collected was \$310. How many of each type of ticket was sold?

12) At a local ball game the hotdogs sold for \$2.50 each adn the hamburbers sold for \$2.75 each. There were 131 total sandwiches sold for a total value of \$342. How many of each sandwich was sold?

13) At a recent Vikings game \$445 in admission tickets was taken in. The cost of a student ticket was \$1.50 adn the cost of a non-student ticket was \$2.50. A total of 232 tickets were sold. How many students and how many non-students attented the game?

14) A bank contains 27 coins in dimes and quarters. The coins have a total value of \$4.95. Find the number of dimes and quarters in the bank.

15) A coin purse contains 18 coins in nickels and dimes. The coins have a total value of \$1.15. Find the number of nickels and dimes in the coin purse.

16) A business executive bought 40 stamps for \$9.60. The purchase included 25¢ stamps and 20¢ stamps. How many of each type of stamp were bought?

17) A postal clerk sold some 15¢ stamps and some 25¢ stamps. Altogether, 15 stamps werw sold for a total cost of \$3.15. How many of each type of stamps were sold?

18) A drawer contains 15¢ stamps and 18¢ stamps. The number of 15¢ stamps is four less than three times the number of 18¢ stamps. The total value of all the stamps is \$1.29. How many 15¢ stamps are in the drawer?

19) The toal value of dimes and quarters in a bank is \$6.05. There are six more quarters than dimes. Find the number of each type of coin in the bank.

20) A child's piggy bank contains 44 coins in quarters and dimes. The coins have a total value of \$8.60. Find the number of quaters in the bank.

21) A coin bank contains nickels and dimes. The number of dimes is 10 less than twice the number of nickels. The total value of all the coins is \$2.75. Find the number of each type of coin in the bank.

22) A total of 26 bills are in a cash box. Some of the bills are one dollar bills, and the rest are five dollar bills. The total amount of cash in the box is \$50. Find the number of each type of bill in the cash box.

23) A bank teller cashed a check for \$200 using twenty dollar bills and ten dollar bills. In all, twelve bills were handed to the customer. Find the number of twenty dollar bills and the number of ten dollar bills.

24) A collection of stamps consists of 22¢ stamps and 40¢ stamps. The number of 22¢ stamps is three more than four times the number of 40¢ stamps. The total value of the stamps is \$8.34. Find the number of 22¢ stamps in the collection.

25) A total of \$27000 is invested, part of it at 12% adn the rest at 13%. The total interst after one year is \$3385. How much was invested at each rate?

26) A total of \$50000 is invested, part of it at 5% adn the rest at 7.5%. The total interest after one year is \$3250. How much was invested at each rate?

27) A total of \$9000 is invested, part of it at 10% and the rest at 12%. The total interest after one year is \$1030. How much was invested at each rate?

28) A total of \$18000 is invested, part of it at 6% and the rest at 9%. The total interest after one year is \$1248. How much was invested at each rate?

29) An inheritance of \$10000 is invested in 2 ways, part at 9.5% and the remainder at 11%. The combined annual interest was \$1038.50. How much was invested at each rate?

30) Kerry earned a total of \$900 last year on his investments. If \$7000 was invested at a certain rate of return and \$9000 was invested in a fund with a rate that was 2% higher, find the two rates of interest.

31) Jason earned \$256 interest last year on his investments. If \$1600 was invested at a certain rate of return and \$2400 was invested in a fund with a rate that was double the rate of the first fund, find the two rates of interest.

32) Millicent earned \$435 last yaer in interest. If \$3000 was invested at a certain rate of return and \$4500 was invested in a fund with a rate that was 2% lower, find the two rates of interest.

33) A total of \$85000 is invested, part of it at 6% and the rest at 3.5%. The total interest after oen year is \$385. How muhcw as invested at each rate?

34) A total of \$12000 was invested, part of it at 9% and the rest at 7.5%. The total interest after one year is \$1005. How much was invested at each rate?

35) A total of \$15000 is invested, part of it at 8% and the rest at 11%. The total interest after one year is \$1455. How muhc was invested at each rate?

36) A total of \$17500 is invested, part of it at 7.25% and the rest at 6.5%. The total interest after one year is \$1227.50. How much was invested at each rate?

37) A total of \$6000 is invested, part of it at 4.25% and the rest at 5.75%. The total interest after one year is \$1217.50. How much was invested at each rate?

38) A total of \$14000 is invested, part of it at 5.5% and the rest at 9%. The total interest after on year is \$910. How much was invested at each rate?

39) A total of \$11000 is invested, part of it at 6.8% and the rest at 8.2%. The total interest after one year is \$797. How much was invested at each rate?

40) An investment portfolio earned \$2010 in interest last year. If \$3000 was invested at a certain rate of return and \$24000 was invested in a fund with a rate that was 4% lower, find the two rates of interest.

41) Samantha earned \$1480 interest last year on her investments. If \$5000 was invested at a certain rate of return and \$11000 was invested in a fund with a rate that was two-thirds the rate of the first fund, find the two rates of interest.

42) A man has \$5.10 in nickels, dimes, and quarters. There are twice as many nickels as diems and 3 more dimes than quarters. How many coins of each kind were there?

43) 30 coins having a value of \$3.30 consists of nickels, dimes and quarters. If there are 40 coins in all and 3 times as many dimes as quarters, how many coins of each kind were there?

44) A bag contains nickels, dimes and quarters having a value of \$3.75. If there are 40 coins in all and 3 times as many dimes as quarters, how many coins of each kind were there?

4.6

Systems of Equations - Mixture Problems

One application of systems of equations are mixture problems. Mixture problems are ones where two different solutions are mixed together resulting in a new final solution. We will use the following table to help us solve mixture problems:

	Amount	Part	Total
Item 1			
Item 2			
Final			

The first column is for the amount of each item we have. The second column is labled “part”. If we mix percentages we will put the rate (written as a decimal) in this column. If we mix prices we will put prices in this column. Then we can multiply the amount by the part to find the total. Then we can get an equation by adding the amount and/or total columns that will help us solve the problem and answer the questions.

These problems can have either one or two varaibles. We will start with one variable problems.

Example 27.

A chemist has 70 mL of a 50% methane solution. How much of a 80% solution must she add so the final solution is 60% methane?

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final			

Set up the mixture table. We start with 70, but don't know how much we add, that is x. The part is the percentages, 0.5 for start, 0.8 for add.

	Amount	Part	Total
Start	70	0.5	
Add	x	0.8	
Final	$70+x$	0.6	

Add amount column to get final amount. The part for this amount is 0.6 because we want the final solution to be 60% methane.

	Amount	Part	Total
Start	70	0.5	35
Add	x	0.8	$0.8x$
Final	$70+x$	0.6	$42+0.6x$

Multiply amount by part to get total.
be sure to distribute on the last row: $(70+x)0.6$

$35+0.8x=42+0.6x$ The last column is our equation by adding

$\underline{-0.6x \qquad -0.6x}$ Move variables to one side, subtract $0.6x$

$35+0.2x=42$ Subtract 35 from both sides

$\underline{-35 \qquad -35}$

$0.2x=7$ Divide both sides by 0.2

$\overline{0.2} \quad \overline{0.2}$

$x=35$ We have our x!

35 mL must be added Our Solution

The same process can be used if the starting and final amount have a price attached to them, rather than a percentage.

Example 28.

A coffee mix is to be made that sells for \$2.50 by mixing two types of coffee. The cafe has 40 mL of coffee that costs \$3.00. How much of another coffee that costs \$1.50 should the cafe mix with the first?

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final			

Set up mixture table. We know the starting amount and it's cost, \$3. The added amount we do not knwo but we do know its cost is \$1.50.

	Amount	Part	Total
Start	40	3	
Add	x	1.5	
Final	$40+x$	2.5	

Add the amounts to get the final amount.
We want this final amount to sell for \$2.50.

	Amount	Part	Total
Start	40	3	120
Add	x	1.5	$1.5x$
Final	$40+x$	2.5	$100+2.5x$

Multiply amount by part to get the total.
Be sure to distribute on the last row $(40+x)2.5$

$$120+1.5x=100+2.5x$$ Adding down the total column gives our equation

$$\underline{-1.5x \qquad -1.5x}$$ Move variables to one side by subtracting $1.5x$

$$120=100+x$$ Subtract 100 from both sides

$$\underline{-100-100}$$

$$20=x$$ We have our x.

20mL must be added. Our Solution

The above problems illustrate how we can put the mixture table together and get an equation to solve. However, here we are interested in systems of equations, with two unknown values. The following example is one such problem.

Example 29.

A farmer has two types of milk, one that is 24% butterfat and another which is 18% butterfat. How much of each should he use to end up 42 gallons of 20% butterfat?

	Amount	Part	Total
Milk 1	x	0.24	
Milk 2	y	0.18	
Final	42	0.2	

We don't know either start value, but we do know final is 42. Also fill in part column with percentage of each type of milk including the final solution

	Amount	Part	Total
Milk 1	x	0.24	$0.24x$
Milk 2	y	0.18	$0.18y$
Final	42	0.2	8.4

Multiply amount by part to get totals.

$$x+y=40$$ The amount column gives one equation

$$0.24x+0.18y=8.4$$ The total column gives *a* second equation.

$$
\begin{array}{rl}
-0.18(x+y)=(40)(-0.18) & \text{Use addition. Multiply first equation by } -0.18 \\
-0.18x-0.18y=-7.2 & \\
& \\
-0.18x-0.18y=-7.2 & \text{Add the equations together} \\
\underline{0.24x+0.18y=8.4} & \\
0.06x \qquad\quad =1.2 & \text{Divide both sides by } 0.06 \\
\overline{0.06} \qquad\quad \overline{0.06} & \\
x=20 & \text{We have our } x\text{, 20 gal of 24\% butterfat} \\
(20)+y=42 & \text{Plug into origional equation to find } y \\
\underline{-20 \qquad -20} & \text{Subtract 20 from both sides} \\
y=22 & \text{We have our } y\text{, 22 gal of 18\% butterfat} \\
\text{20 gal of 24\% and 22 gal of 18\%} & \text{Our Solution}
\end{array}
$$

The same process can be used to solve mixtures of prices with two unknowns.

Example 30.

In a candy shop, chocolate which sells for \$4 a pound is mixed with nuts which are sold for \$2.50 a pound are mixed to form a chocolate-nut candy which sells for \$3.50 a pound. How much of each are used to make 30 pounds of the mixture?

	Amount	Part	Total
Chocolate	c	4	
Nut	n	2.5	
Final	30	3.5	

Using our mixture table, use c and n for variables
We do know the final amount (30) and price, include this in the table

	Amount	Part	Total
Chocolate	c	4	$4c$
Nut	n	2.5	$2.5n$
Final	30	3.5	105

Multiply amount by part to get totals

$$
\begin{array}{rl}
c+n=30 & \text{First equation comes from the first column} \\
4c+2.5n=105 & \text{Second equation comes from the total column} \\
& \\
c+n=30 & \text{We will solve this problem with substitution} \\
-n \quad -n & \text{Solve for } c \text{ by subtracting } n \text{ from the first equation} \\
c=30-n &
\end{array}
$$

$$
\begin{array}{rl}
4(30-n)+2.5n=105 & \text{Substitute into untouched equation}\\
120-4n+2.5n=105 & \text{Distribute}\\
120-1.5n=105 & \text{Combine like terms}\\
\underline{-120 \qquad -120} & \text{Subtract 120 from both sides}\\
-1.5n=-15 & \text{Divide both sides by } -1.5\\
\overline{-1.5} \quad \overline{-1.5} & \\
n=10 & \text{We have our } n\text{, 10 lbs of nuts}\\
c=30-(10) & \text{Plug into } c= \text{ equation to find } c\\
c=20 & \text{We have our } c\text{, 20 lbs of chocolate}\\
\text{10 lbs of nuts and 20 lbs of chocolate} & \text{Our Solution}
\end{array}
$$

With mixture problems we often are mixing with a pure solution or using water which contains none of our chemical we are interested in. For pure solutions, the percentage is 100% (or 1 in the table). For water, the percentage is 0%. This is shown in the following example.

Example 31.

A solution of pure antifreeze is mixed with water to make a 65% antifreeze solution. How much of each should be used to make 70 L?

	Amount	Part	Final
Antifreeze	a	1	
Water	w	0	
Final	70	0.65	

We use a and w for our variables. Antifreeze is pure, 100% or 1 in our table, written as a decimal. Water has no antifreeze, its percentage is 0. We also fill in in final percent

	Amount	Part	Final
Antifreeze	a	1	a
Water	w	0	0
Final	70	0.65	45.5

Multiply to find final amounts

$$
\begin{array}{rl}
a+w=70 & \text{First equation comes from first column}\\
a=45.5 & \text{Second equation comes from second column}\\
(45.5)+w=70 & \text{We have } a\text{, plug into to other equation}\\
-45.5 \qquad -45.5 & \text{Subtract 45.5 from both sides}\\
w=24.5 & \text{We have our } w\\
45.5\,L \text{ of antifreeze and } 24.5\,L \text{ of water} & \text{Our Solution}
\end{array}
$$

Practice - Mixture Problems

Solve.

1) A tank contains 8000 liters of a solution that is 40% acid. How much water should be added to make a solution that is 30% acid?

2) How much antifreeze should be added to 5 quarts of a 30% mixture of antifreeze to make a solution that is 50% antifreeze?

3) Of 12 pounds of salt water 10% is salt; of another mixture 3% is salt. How many pounds of the second should be added to the first in order to get a mixture of 5% salt?

4) How much alcohol must be added to 24 gallons of a 14% solution of alcohol in order to produce a 20% solution?

5) How many pounds of a 4% solution of borax must be added to 24 pounds of a 12% solution of borax to obtain a 10% solution of borax?

6) How many grams of pure acit must be added to 40 grams of a 20% acid solution to make a solution which is 36% acid?

7) A 100 LB bag of animal feed is 40% oats. How many pounds of oats must be added to this feed to produce a mixture which is 50% oats?

8) A 20 oz alloy of platinum that costs \$220 per ounce is mixed with an alloy that costs \$400 per ounce. How many ounces of the \$400 alloy should be used to make an alloy that costs \$300 per ounce?

9) How many pounds of tea that cost \$4.20 per pound must be mixed with 12 lb of tea that cost \$2.25 per pound to make a mixture that costs \$3.40 per pound?

10) How many liters of a solvent taht costs \$80 per liter must be mixed with 6 L of a solvent that costs \$25 per liter to make a solvent that costs \$36 per liter?

11) How many kilograms of hard candy that cost \$7.50 per kilogram must be mixed with 24 kg of jelly beans that cost \$3.25 per kilogram to make a mixture that sells for \$4.50 per kilogram?

12) How many kilograms of soil supplement that costs \$7.00 per kilogram must be mixed with 20 kg of aluminum nitrate that costs \$3.50 per kilogram to make a fertilizer that costs \$4.50 per kilogram?

13) How many pounds of lima beans that cost 90¢ per pound must be mixed with 16 lb of corn that cost 50¢ per pound to make a mixture of vegetables that costs 65¢ per pound?

14) How many liters of a blue dye that costs \$1.60 per liter must be mixed with 18 L of anil that costs \$2.50 per liter to make a mixture that costs \$1.90 per liter?

15) Solution A is 50% acid and solution B is 80% acid. How much of each should be used to make 100cc. of a colutoin taht is 68% acid?

16) A certain grade of milk contains 10% butter fat and a certain grade of cream 60% butter fat. How many quarts of each must be taken so as to obtain a mixture of 100 quarts that will be 45% butter fat?

17) A farmer has some cream with is 21% butterfat and some which is 15% butter fat. How many gallons of each must be mixed to produce 60 gallons of cream which is 19% butterfat?

18) A syrup manufacturer has some pure maple syrup and some which is 85% maple syrup. How many liters of each should be mixed to make 150L which is 96% maple syrup?

19) A chemist wants to make 50ml of a 16% acid solution by mixing a 13% acid solution and an 18% acid solution. How many milliliters of each solution should the chemist use?

20) A hair dye is made by blending 7% hydrogen peroxide solution and a 4% hydrogen peroxide solution. How many milliliters of each are used to make a 300 ml solution that is 5% hydrogen peroxide?

21) A paint that contains 21% green dye is mixed wiht a paint that contains 15% green dye. How many gallons of each must be used to make 60 gal of paint that is 19% green dye?

22) A candy mix sells for $2.20 per kilogram. It contains chocolates worth $1.80 per kilogram and other candy worth $3.00 per kilogram. How much of each are in 15 kilograms of the mixture?

23) To make a weed and feed mixture, the Green Thumb Garden Shop mixes fertilizer worth $4.00/lb. with a weed killer worth $8.00/lb. The mixture will cost $6.00/lb. How much of each should be used to prepare 500 lb. of the mixture?

24) A grocer is mixing 40 cent per lb. coffee with 60 cent per lb. coffee to make a mixture worth 54¢ per lb. How much of each kind of coffee should be used to make 70 lb. of the mixture?

25) A grocer wishes to mix sugar at 9 cents per pound with sugar at 6 cents per pound to make 60 pounds at 7 cents per pound. What quantity of each must he take?

26) A high-protein diet supplement that costs $6.75 per pound is mixed with a vitamin supplement that costs $3.25 per pound. How many pounds of each should be used to make 5 lb of a mixture that costs

27) A goldsmith combined an alloy that costs $4.30 per ounce with an alloy that costs

28) A grocery store offers a cheese and fruit sampler that combines cheddar cheese that costs $8 per kilogram with kiwis that cost $3 per kilogram. How many kilograms of each were used to make a 5 kg mixture that costs $4.50 per kilogram?

29) The manager of a garden shop mixes grass seed that is 60% rye grass with 70 lb of grass seed that is 80% rye grass to make a mixture that is 74% rye grass. How much of the 60% mixture is used?

30) How many ounces of water evaporated from 50 oz of a 12% salt solution to produce a 15% salt solution?

31) A caterer made an ice cream punch by combining fruit juice that cost $2.25 per gallon with ice cream that costs $3.25 per gallon. How many gallons of each were used to make 100 gal of punch costing $2.50 per pound?

32) A clothing manufacturer has some pure silk thread and some thread that is 85% silk. How many kilograms of each must be woven together to make 75 kg of cloth that is 96% silk?

33) A carpet manufacturer blends two fibers, one 20% wool and the second 50% wool. How many pounds of each fiber should be woven together to produce 600 lb of a fabric that is 28% wool?

34) How many pounds of coffee that is 40% java beans must be mixed with 80 lb of coffee that is 30% java beans to make a coffee blen that is 32% java beans?

35) The manager of a specialty food store combined almonds that cost $4.50 per pound with walnuts that cost $2.50 per pound. How many pounds of each were used to make a 100 lb mixture that cost $3.24 per pound?

36) How many grams of pure salt must be added to 40 g of a 20% solution to make a saline solution that is 10% salt?

37) How many ounces of dried apricots must be added to 18 oz of a snack mix that contains 20% dried apricots to make a mixture that is 25% dried apricots?

38) How many milliliters of pure chocolate must be added to 150 ml of chocolate topping that is 50% chocolat3e to make a topping that is 75% chocolate?

39) How many ounces of pure bran flakes must be added to 50 oz of cereal that is 40% bran flakes to produce a mixture thatis 50% bran flakes?

40) A ground meat mixture is formed by combining meat that costs $2.20 per pound with meat that costs $4.20 per pound. How many pounds of each were used to make a 50 lb mixture tha costs $3.00 per pound?

41) How many grams of pure water must be added to 50 g of pure acid to make a solution that is 40% acid?

42) A lumber company combined oak wood chips that cost $3.10 per pound with pine wood chips that cost $2.50 per pound. How many pounds of each were used to make an 80 lb mixture costing $2.65 per pound?

43) How many ounces of pure water must be added to 50 oz of a 15% saline solution to make a saline solution that is 10% salt?

44) A tea that is 20% jasmine is blended with a tea that is 15% jasmine. How many pounds of each tea are used to make 5 lb of tea that is 18% jasmine?

Answers - Graphing

1) $\{-1,2\}$
2) $\{-4,3\}$
3) $\{-1,3\}$
4) $\{-3,1\}$
5) No Solution
6) $\{-2,-2\}$
7) $\{-3,1\}$
8) $\{4,4\}$
9) $\{-3,-1\}$
10) No Solution
11) $\{3,-4\}$
12) $\{4,-4\}$
13) $\{1,-3\}$
14) $\{-1,3\}$
15) $\{3,-4\}$
16) No Solution
17) $\{2,-2\}$
18) $\{4,1\}$
19) $\{-3,4\}$
20) $\{2,-1\}$
21) $\{3,2\}$
22) $\{-4,-4\}$
23) $\{-1,-1\}$
24) $\{2,3\}$
25) $\{-1,-2\}$
26) $\{-4,-3\}$
27) No Solution
28) $\{-3,1\}$
29) $\{4,-2\}$
30) $\{1,4\}$

Answers - Solving with 2 Variables - Substitution

1) $\{1,-3\}$
2) $\{-3,2\}$
3) $\{-2,-5\}$
4) $\{0,3\}$
5) $\{-1,-2\}$
6) $\{-7,-8\}$
7) $\{1,5\}$
8) $\{-4,-1\}$
9) $\{3,3\}$
10) $\{4,4\}$
11) $\{2,6\}$
12) $\{-3,3\}$
13) $\{-2,-6\}$
14) $\{0,2\}$
15)$\{1,-5\}$
16)$\{-1,0\}$
17)$\{-1,8\}$
18)$\{3,7\}$
19) $\{2,3\}$
20) $\{8,-8\}$
21) $\{1,7\}$
22) $\{1,7\}$
23) $\{-3,-2\}$
24) $\{1,-3\}$
25) $\{1,3\}$
26) $\{2,1\}$
27) $\{-2,8\}$
28) $\{-4,3\}$
29) $\{4,-3\}$
30) $\{-1,5\}$
31) $\{0,2\}$
32) $\{0,-7\}$
33) $\{0,3\}$
34) $\{1,-4\}$
35) $\{4,-2\}$
36) $\{8,-3\}$
37) $\{2,0\}$
38) $\{2,5\}$
39) $\{-4,8\}$
40) $\{2,3\}$

Answers - Solving with 2 Variables - Elimination

1) $\{-2,4\}$
2) $\{2,4\}$
3) No solution
4) Infinite number of solutions
5) No solution

6) Infinite number of solutions

7) No solution

8) $\{2,-2\}$

9) $\{-3,-5\}$

10) $\{-3,6\}$

11) $\{-2,-9\}$

12) $\{1,-2\}$

13) $\{0,4\}$

14) $\{-1,0\}$

15) $\{8,2\}$

16) $\{0,3\}$

17) $\{4,6\}$

18) $\{-6,-8\}$

19) $\{-2,3\}$

20) $\{1,2\}$

21) $\{0,-4\}$

22) $\{0,1\}$

23) $\{-2,0\}$

24) $\{2,-2\}$

25) $\{-1,-2\}$

26) $\{-3,0\}$

27) $\{-1,-3\}$

28) $\{-3,0\}$

29) $\{-8,9\}$

30) $\{1,2\}$

31) $\{-2,1\}$

32) $\{-1,1\}$

33) $\{0,0\}$

34) Infinite number of solutions

Answers - Solving Equations with three Variables

1) $(1,-1,2)$

2) $(5,-3,2)$

3) $(2,3,-2)$

4) $(3,-2,1)$

5) $(-2,-1,4)$

6) $(-3,2,1)$

7) $(1,2,3)$

8) $\propto$ solutions

9) $(0,0,0)$

10) $\propto$ solutions

11) $(19,0,-13)$

12) $\propto$ solutions

13) $(0,0,0)$

14) $\propto$ solutions

15) $(2,\frac{1}{2},-2)$

16) $\propto$ solutions

17) $(-1,2-3)$

18) $(-1,2,-2)$

19) $(0,2,1)$

20) no solution

21) $(10,2,3)$

22) no solution

23) $(2,3,1)$

24) $\propto$ solutions

25) no solutions

26) $(1,2,4)$

27) $(-25,19,-25)$

28) $(\frac{2}{7},\frac{3}{7},\frac{2}{7})$

29) $(1,-3,-2,-1)$

30) $(7,4,5,6)$

31) $(1,-2,4,-1)$

32) $(-3,-1,0,4)$

Answers - Value Problems

1) 33Q, 70D

2) 26 h, 8 n

3) 236 adult, 342 child

4) 9d, 12q

5) 8, 19

6) 7q, 4h

7) 9, 18

8) 25, 20

9) 203 adults, 226 child

10) 130 adults, 70 students

11) 128 card, 75 no card

12) 73 hotdogs, 58 hamburgers

13) 135 students, 97 non-students

14) 12d, 15q

15) 13n, 5d

16) 8 20¢, 32 25¢

17) 6 15¢, 9 25¢

18) 5

19) 13 d, 10 q

20) 28 q

21) 15 n, 20 d

22) 20 \$1, 6 \$5

23) 8 \$20, 4 \$10

24) 27

25) \$12500 @ 12%
\$14500 @ 13%

26) \$20000 @ 5%
\$30000 @ 7.5%

27) \$2500 @ 10%
\$6500 @ 12%

28) \$12400 @ 6%
\$5600 @ 9%

29) \$4100 @ 9.5%
\$5900 @ 11%

30) \$7000 @ 4.5%
\$9000 @ 6.5%

31) \$1600 @ 4%;
\$2400 @ 8%

32) \$3000 @ 4.6%
\$4500 @ 6.6%

33) \$3500 @ 6%;
\$5000 @ 3.5%

34) \$7000 @ 9%
\$5000 @ 7.5%

35) \$6500 @ 8%;
\$8500 @ 11%

36) \$12000 @ 7.25%
\$5500 @ 6.5%

37) \$3000 @ 4.25%;
\$3000 @ 5.75%

38) \$10000 @ 5.5%
\$4000 @ 9%

39) \$7500 @ 6.8%;
\$3500 @ 8.2%

40) \$3000 @ 11%;
\$24000 @ 7%

41) \$5000 @ 12%
\$11000 @ 8%

42) 12n, 13d, 10q

43) 18, 4, 8

44) 26n, 7d, 7q

Answers - Mixture Problems

1) $\{2666.7\}$

2) $\{2\}$

3) $\{30\}$

4) $\{1, 8\}$

5) $\{5\}$

6) $\{10\}$

7) $\{20\}$

8) $\{16\}$

9) $\{17.25\}$

10) $\{1.5\}$

11) $\{10\}$

12) $\{8\}$

13) $\{9.6\}$

14) $\{36\}$

15) $\{40, 60\}$

16) $\{30, 70\}$

17) $\{40, 20\}$

18) $\{40, 110\}$

19) $\{20, 30\}$

20) $\{100, 200\}$

21) $\{40, 20\}$

22) $\{10, 5\}$

23) $\{250, 250\}$

24) $\{21, 49\}$

25) $\{20, 40\}$

26) $\{2, 3\}$

27) $\{56, 144\}$

28) $\{1.5, 3.5\}$

29) $\{30\}$

30) $\{10\}$

31) $\{75, 25\}$

32) $\{55, 20\}$

33) $\{440, 160\}$

34) $\{20\}$

35) $\{37, 67\}$

36) $\{10\}$

37) $\{1, 2\}$

38) $\{150\}$

39) $\{10\}$

40) $\{30, 20\}$

41) $\{75\}$

42) $\{20, 60\}$

43) $\{25\}$

44) $\{3, 2\}$